#### **5. Competition model**

**5.1. Introduction**

**Object** Coexistence of two biological species

**Object clarification** coexistence – –, that is the the competition model

**Foundation** Phase plane for the system of differential equations

**Aim** Definition of the law of change in the number of both species depending on the conditions of the process

**5.2. Competition model**

**General supposition** There is a negative influence of both species on each other.

 The function *xi=xi*(*t*) describes the number of *i-*th species at the time *t*, *i=*1,2.

Consider two biological species located in a same area. We assume that none of them directly affects the other species. However, consuming the same food, they certainly enter into intense competition between themselves in view of the limited amount of food.

Determine a mathematical model of the considered process. Obviously, the velocity of number change for *i*th species is proportional to the number of this species

 ** (1)

where *ki* is the growth of the number of this species, *i* = 1,2. If the quantity of foods is unbounded, then these coefficients are constant, and we can analyse these equations separatly. If the quantity of foods is bounded, then the growth coefficients are less and depend from the number of population. Note that both species consume the same food. Now we get the following ***system of differential equations***

  (2)

where *ai* is the real coefficient of growth, and *qi* is a food consumption of *i*th species, *i* = 1,2. The formulas (2) are called the ***competition equations***.

For obtaining its uniqie solution, it is necessary to add the initial conditions. Suppose we now the initial numbers of species *x*10 and *x*20. Now we have theinitial conditions

 *xi*(0) = *xi*0, *i=*1,2. (3)

The Caushy problem (2), (3) id called the ***competition model***.

 **5.3. Equilibrium state for the system**

Find the equilibrium position for the system (1). Equating the right-hand sides of these equations to zero, we have two equalities

  (4)

 . (5)

From the equality (4), it follows that *x*1=0 or *a*1*–b*1(*x*1+*x*2)=0. If the first of them is true, then the equality (5) takes the form (*a*2*–b*2*x*2)*x*2=0. Now we can find *x*2=0 or *d*2*–a*2*q*2*x*2=0. Therefore, we have two equilibrium positions *x*1=0, *x*2=0 and *x*1=0, *x*2=*a*2/*b*2. Analogically, from the equality (5), we find *x*2=0 or *a*2*–b*2*x*1=0. If the first of them is true, then the equality (4) takes the form (*a*1*–b*1*x*1)*x*1=0. Now we can have *x*1=0 or *a*1*–b*1*x*1=0. Therefore, we find two equilibrium positions *x*1=0, *x*2=0 and *x*2=0, *x*1=*a*1/*b*1. The first of them was be determined before. Hence, our system has three equilibrium positions

 *x*1=0, *x*2=0; *x*1=0, *x*2=*a*2/*b*2; *x*1=*a*1/*b*1, *x*2=0. (6)

First equilibrium state is trivial that corresponds to the absence of both species. For the second equilibrium position, we have the second species, and the first one is absent. For the third case, we have the inverse situation with presence of the first species and the absence of the second one.

**5.4. Dividing of the phase space for the system**

We consider again the first quadrant, because both state functions are non-negative as numbers of populations. Then the value at the right hand-side of the first equality (2) is positive if the number *a*1*–b*1(*x*1+*x*2) is positive, i.e., *k*1>0; see equality (1). This is true if *x*1+*x*2 is less than *a*1/*b*1. Therefore, the derivative of the first unknown function is positive, and this function increases if the point (*x*1,*x*2) staies lower than the line *x*1+*x*2=*a*1/*b*1, i.e., *k*1=0. Then the function *x*1 decreases if the point (*x*1,*x*2) staies higher than this line, i.e. *k*1<0.

Analogically, the value at the right hand-side of the second equality (2) is positive if the number *a*1*–b*1(*x*1+*x*2) is positive, i.e., *k*2>0; see equality (1). This is true if *x*1+*x*2 is less than *a*2/*b*2. Therefore, the derivative of the second unknown function is positive, and this function increases if the point (*x*1,*x*2) staies lower than the line *x*1+*x*2=*a*2/*b*2, i.e., *k*2=0. The function *x*2 decreases if the point (*x*1,*x*2) staies upper than this line, i.e. *k*2<0.

It is very important that the obtained lines

*x*1+*x*2 = *a*1/*b*1,  *x*1+*x*2 = *a*2/*b*2

are parallel. Therefore, we can have three different divisions of the phase plan by parts with the fixed signs of derivative. It depends of the comparison of the ratios *d*1*/a*1 and *a*2/*b*2.

Suppose *a*1/*b*1>*a*2/*b*2. Then the phase plane can be divided by three parts; see Figure 1. In this case, both functions decrease if the point (*x*1,*x*2) staies higher than both lines. Both functions decrease if this point staies lower than both lines. First finction increases, and the second one decreases if this point staies between these lines.

Suppose *a*1/*b*1<*a*2/*b*2. Then the phase plane can be divided too by three parts; see Figure 2. In this case, both functions again decrease if the point (*x*1,*x*2) staies higher than both lines. Both functions decrease if this point staies lower than both lines. First finction decreases, and the second one increases if this point staies between these lines.

If *a*1/*b*1=*a*2/*b*2, i.e., *k*1=*k*2, then the phase plane can be divided by two parts, see Figure 3. Now both functions decrease if the point (*x*1,*x*2) staies higher than the unique line; and they decrease if this point staies lower than the line.



Figure 1. Directions of the system evolution for a1/b1>a2/b2.



Figure 2. Directions of the system evolution for a1/b1<a2/b2.



Figure 3. Directions of the system evolution for a1/b1=a2/b2.

**5.5. Analysis of the system evolution if *a*1/*b*1>*a*2/*b*2**.

Suppose the following inequality holds *a*1/*b*1>*a*2/*b*2. The law of the system evolution depends from the point of start, i.e., from the initial state of the system. Let both initial state be small enough such that the point of start is lower than the lower line; see Figure 1. In this case, both derivatives are positive, so both functions increase; see Figure 4, curve 1. Over time, the phase curve will reach the lower straight line. At this time, the derivative of the second function becomes zero, while the derivative of the first function remains positive. Thus, the movement in the direction of growth of the second function will stop, while the first function continues to increase. As a result, we find ourselves in the area between two straight lines.

In this region, the second function begins to decrease, while the first function continues to increase. If over time the curve approaches the first coordinate axis, then the value of *x*2, and by virtue of the second equation (2), the function *x*2 too will be arbitrarily small. Thus, the decrease of this function will practically stop while the first function continues to increase. If, over time, the curve approaches a straight-line *k*1=0, then the derivative of will be arbitrarily small. Consequently, the increase in *x*1 will practically stop while the second function continues to decrease. Thus, the only possible outcome of the system is the equilibrium position *x*1=*a*1/*b*1, *x*2=0.

Suppose now both initial state be large enough such that the point of start is upper than the upper line. Then both functions decrease; see Figure 4, curve 2. Over time, the phase curve will reach the upper straight line. At this time, the derivative of the first function becomes zero, while the derivative of the second one remains positive. Thus, the movement in the direction of growth of the first function will stop, while the second function continues to decrease. As a result, we find ourselves again in the area between two straight lines. As we already know, in this case the system reaches an equilibrium position *x*1=*a*1/*b*1, *x*2=0.



Figure 4. Variants of the system evolution for a1/b1>a2/b2.

However, a variant is possible when, in the process of decreasing function *x*2, we do not end up in the middle zone, but immediately find ourselves in an equilibrium position, see Figure 4, curve 2. This case is realized if the initial value of *x*2 was sufficiently small.

Finally, the point of start can be between two lines; see Figure 4, curve 2. It can be true if Figure 4, curve 2. In this case the the first function is monotonically increasing, and the second is monotonically decreasing. This continues until an equilibrium position *x*1=*a*1/*b*1, *x*2=0 is reached.

Thus, in this case, the only outcome will be the value *x*1=*a*1/*b*1, *x*2=0, which turns out to be a stable equilibrium position. The remaining two equilibrium positions are unstable.

**5.6. Analysis of the system evolution if *a*1/*b*1<*a*2/*b*2**.

Suppose the following inequality holds *a*1/*b*1<*a*2/*b*2 that which is the opposite situation to the one discussed earlier. In this case, all the results obtained in the previous version remain in force, but the variables *x*1 and *x*2 change places. Again, there are four options for the evolution of the system depending on the location of the starting point, i.e. the initial state of the system; see Figure 5. However, in any case, over time, the system now reaches an equilibrium position *x*1=0, *x*2=*a*2/*b*2, which turns out to be stable. The remaining two equilibrium positions are unstable.



Figure 5. Variants of the system evolution for a1/b1<a2/b2.

**5.6. Analysis of the system evolution if *a*1/*b*1=*a*2/*b*2**.

If the equality*a*1/*b*1=*a*2/*b*2 is true, then the lines *k*1=0 and *k*2=0 coincide. This case is degenerate, because the first multipliers of the values at the left sides of equalities (4) and (5) coincide two. In this situation, both derivatives of the system (2) can be zero if *x*1=0, *x*2=0 (trivial equilibrium position) of for all non-negative values (*x*1,*x*2) satisfying the equality *x*1+*x*2=0. Thus, there is an infinite number of points (the indicated line segment; see Figure 6), which are the equilibrium positions of the system.

If the initial state is small enough, then the point of start is lower than the considered line. Hence, both derivatives of the state functions are positive, so these functions increase. As the phase curve approaches the indicated straight line, both derivatives in equalities (2) become smaller and smaller. Thus, this straight line is reached in the limit with an unlimited increase in time; see Figure 6, curves 1. The specific value of the equilibrium position on the segment is determined by the starting point.

If the initial state is large enough, then the point of start is upper than the considered line. Hence, both derivatives of the state functions are negative, so these functions decrease. As the phase curve approaches the indicated straight line, both derivatives in equalities (2) become again smaller and smaller. Thus, this straight line is reached in the limit with an unlimited increase in time; see Figure 6, curves 2.

Of course, if the point of start belongs to the considered segment (see Figure 6, position 3), then the system remains in this state all the time because it is an equilibrium position.



Figure 6. Variants of the system evolution for a1/b1=a2/b2.

**5.7. Interpretation of results**

As can be seen from the analysis, for the competition model there are two levels of classification of system evolution options. The first of them is determined by the relationship between the coefficients of the equation and uniquely sets the final result, i.e. the state of equilibrium at which the system reaches. The second level is determined by the initial state of the system and indicates exactly how the system reaches the equilibrium position.

First, let us pay attention to the meaning of the relationships between the coefficients of the equation and the outcome of events determined by it. The coefficients *ai* characterize the rate of reproduction of a species, and this is the *bi* describes the consumption of a given species for food. Therefore, the large value of the ratio *ai*/*bi* means that this species reproduces quickly and consumes little food. Thus, the value *ai*/*bi* characterizes viability of this species. The inequality *a*1/*b*1>*a*2/*b*2 is true if the first has a higher birth rate and lower food requirements compared to the second species. We determined that for this case, the system tends to the equilibrium position *x*1=*a*1/*b*1, *x*2=0. This means that the second species is dying out, because it is weaker than first. Under the condition *a*1/*b*1<*a*2/*b*2 the second species is stronger, and the system tends to the equilibrium position *x*1=0, *x*2=*a*2/*b*2. This means that the second species survives, i.e. the strongest wins again. Finally, for the case *a*1/*b*1=*a*2/*b*2, have the same vitality. Under these conditions, none of them can displace the other, i.e. both types remain. The variants of the system evolution is shown in Figure 7.



Figure 7. Variants of the system evolution in the competition model.

Now we will describe the evolution of the system in each scenario depending on the initial values of the number of species. Suppose *a*1/*b*1>*a*2/*b*2, and the initial numbers of species are small enough that corresponds to the curve 1 of the Figure 4. Since the numbers of both species are quite small, there is enough food for everyone. As a result, the numbers of both species are growing. However, as it grows, the need for food also increases. Over time, a moment comes when there is no longer enough food for everyone, i.e. interspecific competition begins. Under these conditions, the strongest species wins. The second species is becoming extinct.

Let us assume that the initial abundance of both species is sufficiently large, which corresponds to curve 2 in Figure 4. Under these conditions, there is not enough food, as a result of which the numbers of both species decrease. However, as the numbers of both species decline, the need for food gradually decreases. Sooner or later, there comes a time when there is enough food, but not for everyone. Then the stronger species displaces the weaker one.

Curve 3 corresponds to the case when the second species not only has less viability, but also a rather small initial abundance. Under these conditions, it dies out due to lack of food even before the system enters the competition zone.

Finally, curve 4 corresponds to the case when the numbers of both species are neither too large nor too small, so that the system is initially in a competition zone. The stronger species wins, and the numbers of both species change monotonically.

Under the inequality *a*1/*b*1<*a*2/*b*2,the second species turns out to be stronger. Here the same processes are observed as in the first case, only the views change places, see Figure 5.

If we have the equality *a*1/*b*1=*a*2/*b*2,then species have equal viability. If initially the number of both species is small, then there is enough food, and the number of species grows, see Figure 6, curves 1. As they grow, the need for food also increases. As a result, the growth rate slows down and gradually tends to zero. Thus, the system reaches an equilibrium position, which is determined by the ratio between the initial numbers of species. Since none of them can displace the other, the initially existing proportions between the values of their numbers are maintained subsequently.

If initially, the numbers of both species are large, then there is not enough food, and the numbers of species are reduced, see curves 2 in Figure 6. However, as they decrease, the need for food decreases. The numbers of both species are gradually stabilizing. Over time, such a number of species is established that can feed itself under given conditions, and the specific values of the number of species are determined by the initial state of the system.